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DEFORMING HOMOTOPY INVOLUTIONS OF 3-MANIFOLDS TO INVOLUTIONS II

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§1. INTRODUCTION

IN [1] we showed that homotopy involutions of Seifert fiber spaces (which contain incompressible fibered tori) can be deformed to involutions if and only if a certain obstruction vanishes. In this note we extend these results to all closed Haken manifolds. Recall that a Haken manifold is a 3-manifold that is compact, orientable, irreducible and sufficiently large. This extension uses the Splitting Theorem of Jaco and Shalen [2] (and Johannson [3]), Thurston's Uniformization Theorem [4] and the results and techniques of [1]. Also, we take this opportunity to make a correction in the proof of Theorem A of [1].

Suppose that M is a Haken manifold. The homomorphism $\mu\colon \pi_1(M)\to Inn(\pi_1(M))$ is defined by $\mu(\alpha)(x)=\alpha x\alpha^{-1}$ for $x\in\pi_1(M)$. For a homotopy involution g of M we let g_* denote an automorphism of $\pi=\pi_1(M)$ induced by $g(g_*$ is unique up to inner automorphism). In [1] it is shown that the vanishing of the obstruction Obs$(Z_2,$ $\pi_1(M), \psi_g)\in H^3(Z_2; z(\pi_1(M)))$ is equivalent to the existence of an element $\tau\in\mu^{-1}(g_*^2)$ with $g_*(\tau)=\tau$. This condition can fail only if M is a Seifert fiber space.

THEOREM 1. *Let M be a closed Haken manifold. Suppose that g is a map of M to itself such that $g^2\simeq 1$. Then g is homotopic to a PL involution if and only if there exists an element $\tau\in\mu^{-1}(g_*^2)$ with $g_*(\tau)=\tau$.*

The proof proceeds according to the following outline. By the Splitting Theorem, M can be split in a canonical way along incompressible tori into $\sigma(M)=M_1\cup\cdots\cup M_m\cup N_1\cup\cdots\cup N_n$, where each N_i is simple and each M_i is a Seifert fiber space. In the case when M is already simple or a Seifert fiber space, the theorem follows by [1 or 4], respectively. In the general case we first deform g to an involution along the splitting tori and then work with the restriction of g on the components of $\sigma(M)$.

§2. THE COMPONENTS OF $\sigma(M)$

A Haken manifold is said to be simple if every rank-two free abelian subgroup of $\pi_1(M)$ is peripheral. Thurston's Uniformization Theorem gives a hyperbolic structure (or metric) of finite volume on the interior of every simple Haken manifold M whose boundary components are all tori. It is then a consequence of Mostow's Rigidity Theorem that every homotopy involution on M can be homotoped to a PL involution. For the bounded components of $\sigma(M)$ we need to perform this deformation to an involution by a homotopy which is constant on the boundary.

THEOREM 2. *Let M be a Haken manifold that is either simple or a Seifert fiber space. Assume that $\partial M\neq\phi$ and each component of ∂M is a torus. If g is a map of M*

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such that $g^2 \simeq 1$ rel ∂M then g is homotopic to a PL involution by a homotopy that is constant on ∂M.

Proof. If M is a Seifert fiber space this follows directly from Theorem C of[1]. If ∂M is not incompressible then M is a solid torus. Thus, we may suppose that M is a simple Haken manifold with incompressible boundary and that M is not a Seifert fiber space. In addition, by [9] we may assume that g is a homomorphism.

We may assume that M contains a collar neighborhood $U = \partial M \times [0, 1]$ with $\partial M = \partial M \times \{0\}$ such that $g(U) = U$ and such that the homotopy G: $g^2 \simeq 1$ carries U to itself at each stage. Let $M' = M \setminus \mathring{U}$. By Thurston's Theorem there is an involution h' of M' homotopic to $g \mid M'$. It follows from Theorem 7.1(b) of [9] that h' is in fact isotopic to $g \mid M'$ in the present case, since line bundles over tori and Klein bottles are Seifert fiber spaces and have been excluded. Extend this isotopy to a homotopy constant on ∂M from g to a map $h : M \to M$, so that now $g \simeq h$ rel ∂M, $h' = h \mid M'$ and $h \mid \partial M$ are involutions, and $h^2 \simeq 1$ rel ∂M by a homotopy G with $G(U \times I) = U$. It now suffices to show that $h'' = h \mid U$ is homotopic to an involution by a homotopy constant on ∂U.

Choose a base point $x_0 \in \partial M \times \{1\}$. Then $h^2(x_0) = x_0$ and the trace τ of the cyclic homotopy $G \mid M' \times I : h'^2 \simeq 1$ represents an element of the center of $\pi_1(M')$, hence $\tau \simeq 0$ rel x_0. By ([1], (3.1)) we can assume that $\tau = x_0$. Therefore, restricting the homotopy G to U, we have that $h'' \mid \partial M \times \{1\}$ is an involution and $G' : h''^2 \simeq 1$ rel $(\{x_0\} \cup \partial M)$ with $G'_t(\partial M \times \{1\}) = \partial M \times \{1\}$. Now it follows from ([1], (3.4)) that there is a homotopy $h''^2 = 1$ that is constant on ∂U, and then from Theorem C [1], that h'' is homotopic rel ∂U to an involution.

§3. PROOF OF THEOREM

Proof. By [9] we may assume that g is a homeomorphism of M. By the Splitting Theorem there exists a system F of incompressible tori with $\sigma(M) = M_1 \cup \cdots \cup M_m \cup N_1 \cup \cdots \cup N_n$, where each M_i is simple and each N_i is a Seifert fiber space. Furthermore there is an isotopy carrying $g(F)$ to F. Thus we may assume, after an isotopy of g, that $g(F) = F$. For each component F_i of F we have either $g(F_i) = F_i$ or $g(F_i) \cap F_i = \phi$.

Let $G : g^2 \simeq 1$ be a homotopy. Then by the argument in §3 of [1], which carries through for these more general 3-manifolds, we may assume that $(g \mid F)^2 = 1$ and $G : g^2 \simeq 1$ rel F.

Case 1. $m + n > 1$. Let $g_i = g \mid M_i$, $g'_j = g \mid N_j$. Since $G : g^2 \simeq 1$ rel F induces a homotopy of $g_i^2 \simeq 1$ rel ∂M_i and $g_j'^2 \simeq 1$ rel ∂N_j, we can apply Theorem 2 to deform each g_i, g'_j to an involution by a homotopy constant on F. This gives the desired deformation of g to an involution.

Case 2. $M = M_1$. This is Thurston's Theorem.

Case 3. $M = N_1$. If M contains an incompressible fibered torus then the theorem in this case follows from Theorem A of [1].

Thus we may assume that M is a closed, sufficiently large, orientable, irreducible Seifert fiber space that does not contain an incompressible fibered torus. Consider the projection $p : M \to B$ to the orbit space B. Suppose γ is a two-sided simple closed curve in B with the property that if γ bounds a 2-cell in B then such a 2-cell contains the images of at least two exceptional fibers. Then $p^{-1}(\gamma)$ is an incompressible fibered torus in M. Since it is assumed that M does not contain any incompressible fibered tori, and yet M is sufficiently large, it is easy to see that the only possibility is for B to be a 2-sphere and M to have exactly three exceptional fibers. By hypothesis, M contains an incompressible two-sided surface S which we have assumed cannot be fibered in M and therefore can not carry the nontrivial center of $\pi_1(M)$. It follows that M can be fibered over S^1 with S as fiber.

We first consider the case when S is a torus. By the next proposition, we can apply the argument of Case 1, replacing F by one of these torus fibers.

PROPOSITION 1. *Let M be a torus bundle over S^1 and g a homeomorphism of M such that $g^2 \simeq 1$. Then there is a fibering of M over S^1 with fiber a torus T' such that $g(T') \simeq T'$.*

Proof. Let $M = T \times R/\phi$, where $T = S^1 \times S^1$. By Lemma 3.1 of [5] there exists a splitting of $H_1(M) = H + Z$ such that $g_*(H) = H$. Let h denote the composition of $\pi_1(M) \to H_1(M) = H + Z \to Z$. If we let $K = \ker h$ then $g_\#(K) = K$. Since $\pi_1(M)$ is polycyclic, all its subgroups, and in particular K, are also polycyclic and hence finitely generated. Thus we can apply Stallings' fibering theorem [8] to conclude that M fibers over S^1 with fiber a closed orientable surface T', where $\pi_1(T') = K$. Since K is polycyclic, T' must be a torus. Moreover, $g_\#(K) = K$ implies that $g(T') \simeq T'$.

Now consider the case when S has a negative Euler characteristic. It follows from ([1], Lemma 6.2) that we may assume $g(S) = S$. Choose a basepoint $x_0 \in S$ and deform g such that $g(x_0) = x_0$. The desired result follows from the next proposition.

PROPOSITION 2. *Let* $M = S \times R^1/\phi$, where *S is a closed, orientable surface of negative Euler characteristic. Suppose that g is a homeomorphism of M such that $g([S \times 0]) = [S \times 0]$. Assume that there exists a homotopy $H : g^2 \simeq 1$ whose trace represents an element $\tau \in \pi_1(M, x_0)$ for which $g_*(\tau) = \tau$. Then there exists a homeomorphism h homotopic to g such that $h^2 = 1$.*

Proof. This is an immediate corollary to the proof of Theorem 5 in [7]. The only case needing comment is that in which g interchanges the sides of $[S \times 0]$. From the fact that g interchanges the sides of $[S \times 0]$ and $g_*(\tau) = \tau$, it follows that $\tau \in \pi_1(S, x_0)$. Thus, we clearly have $\deg F \cdot H \mid \Sigma = 0$ which is, as we indicate in ([7], Erratum), all that is necessary to obtain the conclusion in this case.

§4. A CORRECTION

We correct the proof of Theorem A in [1]. The conclusion that the involution can be chosen to be fiber preserving should be dropped in the cases (a)–(c) below. Our claim in §5 that we may assume g to be fiber-preserving does not apply in the following three cases:

(a) $M = S^1 \times S^1 \times S^1$

(b) M is the orientable S^1-bundle over the Klein bottle

(c) M is the Seifert fiber space obtained from $S^1 \times S^1 \times [-1, 1]$ by identifying $(x, y, -1) \sim (-x, \bar{y}, -1)$ on one end and $(x, y, 1) \sim (\bar{x}, -y, 1)$ on the other end. (M can also be obtained from two copies of a twisted I-bundle over the Klein bottle by identifying the two tori boundary components via an involution interchanging the factors).

In each of these cases our proof of Theorem A in ([1], §5) applies (except for the fiber preserving conclusion), since in order to use §3 of [1] it is sufficient to find an incompressible fibered torus F so that $g(F)$ is isotopic to F. To construct such a torus F, observe that in case (b) M can be viewed as a torus bundle over S^1, $M = S^1 \times S^1 \times [0, 1]/\phi$, where $\phi(x, y) = (\bar{x}, \bar{y})$. Thus in cases (a) and (b) the existence of F follows from Proposition 1. (It is easy to see that F can be chosen to be fibered.) In case (c), let $M = S^1 \times S^1 \times [-1, 1]/\sim$ as described and let $T = S^1 \times S^1 \times 0$, $S = S^1 \times \{x_1, x_2, x_3, x_4\} \times [-1, 1]/\sim$, and $U = \{x_1, x_2, x_3, x_4\} \times S^1 \times [-1, 1]/\sim$, where $x_k = \exp((2k-1)\pi i/4)$. It can be shown [6] that S, T and U are equivalently embedded fibered incompressible tori, but no pair are isotopic. Furthermore, any incompressible torus in M is isotopic to one of these three tori. Now if $g(T)$ is isotopic to T we are done. If $g(T)$ is isotopic to S then $g(S)$ is isotopic to $g(g(T))$ which is isotopic to T since $g^2 \simeq 1$. It follows that $g(U)$ must be isotopic to U and we can take $F = U$. In the case when $g(T)$ is isotopic to U we can take $F = S$.

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